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Geometrical theory of dislocations in bodies with microstructure

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Abstract

A material body with smoothly distributed microstructure can be seen geometrically as a fibration or, when the symmetry group is specified, as a fiber bundle. Within this very general framework, we present a geometric description of such material bodies in terms of fiber jets. We introduce the notion of fiber frame and construct the corresponding Lie groupoid and fiber G-structure. Then, physical properties of a material body with microstructure as uniformity and homogeneity can be translated in geometrical terms as transitivity for the Lie groupoid or integrability for the fiber G-structure.
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1. Introduction

Modern theories of uniformity and homogeneity of material bodies consider the body B as the base manifold of a fiber bundle. For materially simple bodies, this fiber bundle has been usually taken to be the tangent bundle or the principal frame bundle by Kondo [7] and his collaborators in Japan, followed by Eshelby, Bilby, Kroener [8] and others in Europe. A rigorous geometric formulation, within the context of continuum mechanics, for concepts as uniformity and homogeneity of a materially simple body has been given by Noll [15] and Wang [17]. The formalism introduced by these authors has been followed by many generalizations that comprise theories of higher-grade materials [9–11] and generalized Cosserat

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bodies, [2,3,5,6]. In these generalizations the material body is viewed as a classical body endowed with some kind of internal structure (or microstructure). Whereas for materially simple bodies the fiber-bundle formalism was used merely as a mathematical substratum to allow for the definition of so-called *material connections* and other such geometrical quantities, in the case of bodies with internal structure it is the body itself that needs to be regarded as a fiber bundle, whose typical fiber carries the microstructural information. To develop a geometric theory of uniformity and homogeneity from this concept, it becomes necessary, therefore, to consider further bundles whose base manifold is itself a fiber bundle. In this paper we propose a general theory whereby the nature of the typical fiber of the body bundle is not specified a priori. Previous attempts in the same direction have been carried out in [1,4].

In the first part of this paper we introduce and study some geometric objects such as fiber jets, the fiber Lie groupoid, the fiber frame bundle and the fiber G-structure. First section in part I is a general overview on fibrations and fiber bundles, as it has been presented in [13,14]. The concept of fiber jet of a fiber bundle, which is the mathematical background in our paper, is introduced in Section 2.2, following some work developed in [16]. Within this very general framework, we study in Section 2.3 the Lie groupoid one can associate to a fiber jet, following the general theory on Lie groupoids from [12]. In the last section of first part we introduce the concept of fiber G-structure and determine necessary and sufficient conditions for the integrability of such structure. All geometric objects introduced in the first part are related in second part with material properties of a body with microstructure. In this direction we prove that the uniformity of a body with microstructure translates into the fact that associated groupoid is a Lie groupoid, while the homogeneity corresponds to the integrability of a fiber G-structure. The correspondence between the two parts of the paper follows the outline of the theory developed in [1]. For the particular case when the microstructure is linear the theory developed in this paper reduces to the theory of second order materials developed in [2,10].

2. Part I. Geometric background on fiber jets

2.1. Fibrations and fiber bundles

Let B and F be two differentiable manifolds of dimensions n and m, respectively, and let M be a nonempty set and $\pi : M \to B$ is surjective map. The collection (M, π, B, F) is said to be a *fibration* if the following conditions are satisfied:

(i) B can be covered by a family of open sets $U, V, W, \ldots$, such that for every open set U of the family, there exists a bijection $\tau_U : \pi^{-1}(U) \to U \times F$ that makes the following diagram commutative:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tau_U} & U \times F \\ & \searrow^{\pi} & \downarrow{pr_1} \\ & & U \end{array}$$

(ii) If $x \in U \cap V$, $\tau_U : \pi^{-1}(U) \to U \times F$ and $\tau_V : \pi^{-1}(V) \to V \times F$, then $\tau_{V,x} \circ \tau_{U,x}^{-1}$ is a diffeomorphism of the manifold F.

Here $\tau_{U,x}$ is the restriction of the bijection τ_U to the fiber $\pi^{-1}(x)$. Consequently, $\tau_{U,x} : \pi^{-1}(x) \to F$ is a bijection, too. The pairs (U, τ_U) are called fibered charts.

Next we shall see that this definition allows us to define a C^∞ structure on M such that the map π is a submersion and each fiber $\pi^{-1}(x)$ is an imbedded submanifold of M (see [14, Chapter 1]).

Consider $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ a smooth atlas of the manifold B such that the covering $\{U_\alpha\}_{\alpha\in I}$ is finer than the covering $U, V, W, \ldots$ from (i), and let $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ be an atlas of the smooth manifold F. Then a base for a topology on M is given by the sets

$$W_{\alpha\beta} = \tau_U^{-1}(U_\alpha \times V_\beta) \subset \pi^{-1}(U),$$

where U_α is such that $U_\alpha \subset U$. With respect to this topology on M, the bijection $\tau_U : \pi^{-1}(U) \to U \times F$ is a homeomorphism. If we denote by $\tau_U^{\alpha\beta}$ the restriction of τ_U to $W_{\alpha\beta}$, then $\Phi_{\alpha\beta} = (\phi_\alpha \times \psi_\beta) \circ \tau_U^{\alpha\beta}$ is a homeomorphism from $W_{\alpha\beta}$ to an open set of $\mathbb{R}^{n+m}$. This way we have a smooth atlas $\{(W_{\alpha\beta}, \Phi_{\alpha\beta})\}_{(\alpha,\beta)\in I\times J}$ on M, with respect to which M is a differentiable manifold of dimension $n + m$. Then the mappings $\tau_U, \tau_V, \ldots$ are diffeomorphisms, $\pi : M \to B$ is a smooth submersion and the smooth structure of M is unique with these properties.

For the fibration (M, π, B, F), the manifold M is called the total space, B is the base manifold, F the typical fiber and π the canonical submersion. For every $x \in B$, $\pi^{-1}(x)$ is an imbedded submanifold of M that is diffeomorphic to F and it is called the fiber at x, sometimes denoted by F_x. The trivial example of a fibration is given by $(B \times F, pr_1, B, F)$. In the second part of this paper we shall consider a special fibration (M, π, B, F), where the base manifold B has a global chart. We shall assume also that the total space M of the fibration is diffeomorphic to the trivial fibration $B \times F$, which means that M is globally trivializable. But while the trivial fibration has a particular singled-out trivialization, a globally trivializable fibration does not.

From now on, we adopt the convention that the indices $i, j, k, l, \ldots$ vary within the range $1, \ldots, n$, while the indices $a, b, c, d, \ldots$ vary within the range $1, \ldots, m$. The local coordinates on B are denoted by (x^i), the local coordinates on F are denoted by (y^a) and consequently the induced local coordinates on the total space M are (x^i, y^a). With respect to these, the submersion $\pi : M \to B$ has the equations $\pi : (x^i, y^a) \mapsto (x^i)$. A change of local coordinates $(x^i, y^a) \to (\tilde{x}^i, \tilde{y}^a)$ around a point $p \in M$ is given by a set of $n + m$ smooth functions:

$$\tilde{x}^i = \tilde{x}^i(x^j), \quad \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \qquad \tilde{y}^a = \tilde{y}^a(x^j, y^b), \quad \text{rank}\left(\frac{\partial \tilde{y}^a}{\partial y^b}\right) = m. \tag{2.1}$$

For every fixed $(x^j) \in B$, $\tilde{y}^a = \tilde{y}^a(x^j, y^b)$ is the local representation of the diffeomorphism $\tau_{U,x} \circ \tau_{V,x}^{-1}$ of the typical fiber F.

We may use Eqs. (2.1) as the equations of a fibered map $\tau_U : \pi^{-1}(U) \to U \times F$ from a fibered chart (U, τ_U). As it is known, see [13], for a smooth, finite dimensional manifold F, the group $\text{Diff}^\infty(F)$ of all diffeomorphisms of F is an open subset of the infinite dimensional manifold $C^\infty(F, F)$ of all smooth maps of F, with the C^∞-Whitney topology. Consequently, $\text{Diff}^\infty(F)$ is an open submanifold of $C^\infty(M, M)$, where the composition and the inverse map are smooth maps. Then for the infinite dimensional Lie group $\mathrm{Diff}^{\infty}(F)$ its Lie algebra is the vector space $\mathcal{X}_c(F)$ of all smooth vector fields on F with compact support.

Consider now a Lie group $\mathcal{G}$, which is a Lie subgroup of the Lie group of *fiber diffeomorphisms* $\mathrm{Diff}^{\infty}(F)$. A fibration (M, π, B, F) is said to be a *fiber bundle* with structural group $\mathcal{G}$ (we denote it by $(M, \pi, B, F, \mathcal{G})$) if:

(iii) The diffeomorphisms $\tau_{U,x} \circ \tau_{V,x}^{-1}$ of F belong to $\mathcal{G}$ and the map $g_{VU} : V \cap U \to \mathcal{G}$, given by $g_{VU}(x) = \tau_{U,x} \circ \tau_{V,x}^{-1}$ is smooth for every two fibered charts (U, τ_U), (V, τ_V) for which $U \cap V \neq \emptyset$.

For a fiber bundle, the structural group $\mathcal{G}$ acts transitively on the typical fiber F through the left translation $L_g : y \in F \mapsto L_g(y) = gy \in F\, \forall g \in \mathcal{G}$. With these notations, the last m equations of (2.1) can be written now as $\tilde{y}^a = L_g(y^b)$.

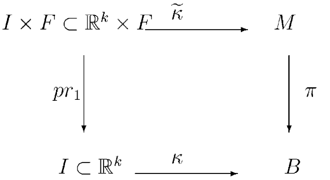
If for a fiber bundle $(M, \pi, B, F, \mathcal{G})$ the typical fiber F coincides with the structural group $\mathcal{G}$, then the fiber bundle is called a *principal fiber bundle*.

2.2. Fiber jets

Let us consider now a fibration (M, π, B, F) and the trivial fibration $(\mathbb{R}^k \times F, \mathrm{pr}_1, \mathbb{R}^k, F)$. For every $x \in B$ and $k \in \{1, \ldots, n\}$, denote by $C_{k,x}(M)$ the set of all differentiable fiber morphisms $\tilde{\kappa}$ from a neighborhood $I \times F$ of $\{0\} \times F$ in $\mathbb{R}^k \times F$ into a neighborhood of $\pi^{-1}(x)$ in M such that:

(i) $\tilde{k}(\{0\} \times F) = \pi^{-1}(x)$.
(ii) The restriction of $\tilde{\kappa}$ to $\{\xi\} \times F$ is a diffeomorphism, $\forall \xi \in I \subset \mathbb{R}^k$.

If a fiber morphism $\tilde{\kappa} : I \times F \to M$ belongs to $C_{k,x}(M)$, then we denote by $\kappa : I \subset \mathbb{R}^k \to B$ the projected map of $\tilde{\kappa}$. This means that the following diagram is commutative: We have also that $\kappa(0) = x$ and $\tilde{k}(\xi, \cdot)$ is a diffeomorphism from $\{\xi\} \times F$ to $\pi^{-1}(\kappa(\xi))$. Then



for any fibered chart (U, τ_U) at $x \in B$, the map $\tau_{U,x} \circ \tilde{k}(0, \cdot)$ is a diffeomorphism of F. This is equivalent to say that there is a fibered chart (U, τ_U) at $x \in B$ such that $\tau_{U,x} \circ \tilde{\kappa}(0, \cdot)$ is the identity of F. This way we have a map $\tau_{U,x} \circ \tilde{\kappa}(\cdot, \cdot) : \xi \in I \subset \mathbb{R}^k \mapsto \tau_{U,x} \circ \tilde{\kappa}(\xi, \cdot) \in \mathrm{Diff}^{\infty}(F)$ that passes through Id_F when $\xi = 0$.

Consider now $(M, \pi, B, F, \mathcal{G})$ a fiber bundle with structural group $\mathcal{G}$. A fiber morphism $\tilde{\kappa} \in C_{k,x}(M)$ if in addition we ask that:

(iii) For any fibered chart (U, τ_U) at $x \in B$, the map $\tau_{U,x} \circ \tilde{\kappa}(\xi, \cdot)$ is an element of the structural group $\mathcal{G}$, $\forall \xi \in I$.

We can choose a fibered chart (U, τ_U) at $x \in B$ such that $\tau_{U,x} \circ \tilde{\kappa}(0, \cdot)$ is the neutral element of $\mathcal{G}$. Then, $\tau_{U,x} \circ \tilde{\kappa}(\cdot, \cdot)$ is a map from a neighborhood of 0 in $\mathbb{R}^k$ into $\mathcal{G}$ that passes through the neutral element e of $\mathcal{G}$ when $\xi = 0$.

Within the set $C_{k,x}(M)$ we define an equivalence relation $\sim_{1,x}$ as follows: let $\tilde{\kappa}_1, \tilde{\kappa}_2 \in C_{k,x}(M)$, then $\tilde{\kappa}_1 \sim_{1,x} \tilde{\kappa}_2$ if

$$\tilde{\kappa}_1(0, f) = \tilde{\kappa}_2(0, f) \quad \text{and} \quad \frac{\partial \tilde{\kappa}_1}{\partial \xi^\alpha}(0, f) = \frac{\partial \tilde{\kappa}_2}{\partial \xi^\alpha}(0, f) \quad \forall \alpha \in \{1, \ldots, k\}\ \forall f \in F.$$

Here (ξ^α) are the Cartesian coordinates on $\mathbb{R}^k$.

If $\tilde{\kappa}_1, \tilde{\kappa}_2 \in C_{k,x}(M)$ are $\sim_{1,x}$-equivalent and κ_1, κ_2 are their projections on B then κ_1 and κ_2 determine the same first order jet, that is $\kappa_1(0) = \kappa_2(0) = x$ and $(\partial \kappa_1/\partial \xi^\alpha)(0) = (\partial \kappa_2/\partial \xi^\alpha)(0)$.

An equivalence class of $C_{k,x}(M)$ with respect to $\sim_{1,x}$ will be denoted by $J^1_{k,x}\tilde{\kappa}$ and will be called a *fiber k-vector* at $x \in B$. The set of all fiber k-vectors at $x \in B$ will be denoted by $T^1_{k,x}(M, B)$ and it is called the *fiber k-tangent space* at $x \in B$. For every $x \in B$, we can always find a fibered chart (U, τ_U) in x such that $\tau_{U,x} \circ \tilde{k}(0, \cdot)$ is the identity map of the typical fiber F. Then $(\partial \kappa/\partial \xi^\alpha)(0)$ are k-vectors tangent at x to B, $\tilde{k}(0, \cdot)$ is a diffeomorphism from $\pi^{-1}(x)$ to F and $(\partial \tilde{\kappa}/\partial \xi^\alpha)(0, \cdot)$ is an element of $L(\mathbb{R}^k, \mathcal{X}_c(F))$. With these considerations we have that any fibered chart (U, τ_U) at $x \in B$ induces a bijection between the fiber k-tangent space $T^1_{k,x}(M, B)$ and $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^k, \mathcal{X}_c(F))$.

In coordinates, two fiber morphisms $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ from $C_{k,x}(M)$ are $\sim_{1,x}$-equivalent if $\tilde{\kappa}_1(\xi, f) = (x^i_1(\xi^\alpha), y^a_1(\xi^\alpha, f))$ and $\tilde{\kappa}_2(\xi, f) = (x^i_2(\xi^\alpha), y^a_2(\xi^\alpha, f))$ satisfy

$$x^i_1(0) = x^i_2(0), \qquad y^a_1(0, f) = y^a_2(0, f) \quad \forall f \in F, \qquad \frac{\partial x^i_1}{\partial \xi^\alpha}(0) = \frac{\partial x^i_2}{\partial \xi^\alpha}(0),$$
$$\frac{\partial y^a_1}{\partial \xi^\alpha}(0, f) = \frac{\partial y^a_2}{\partial \xi^\alpha}(0, f) \quad \forall f \in F.$$

Consequently, we can identify a fiber k-vector $J^1_{k,x}\tilde{\kappa}$ with a triple $((\partial x^i/\partial \xi^\alpha)(0), y^a(0, \cdot), (\partial y^a/\partial \xi^\alpha)(0, \cdot))$. Here $(\partial x^i/\partial \xi^\alpha)(0)$, $\alpha \in \{1, \ldots, k\}$ are k-vectors tangent at x to B, $y^a(0, \cdot)$ is a diffeomorphism of F, while $(\partial y^a/\partial \xi^\alpha)(0, \cdot)$ is an element of $L(\mathbb{R}^k, \mathcal{X}_c(F))$.

When (M, π, B) is a fiber bundle with structural group $\mathcal{G}$, we saw that we can always find a fibered chart (U, τ_U) at $x \in B$ such that $\tau_{U,x} \circ y^a(0, \cdot)$ is the neutral element of $\mathcal{G}$. Then we can identify $(\partial y^a/\partial \xi^\alpha)(0, \cdot)$ with k elements of the Lie algebra $L\mathcal{G}$ of the Lie group $\mathcal{G}$. This is equivalent to say that $(\partial y^a/\partial \xi^\alpha)(0, \cdot)$ is an element of $L(\mathbb{R}^k, L\mathcal{G})$. Consequently, a fibered chart (U, τ_U) at $x \in B$ determines a bijection between the fiber k-tangent space at $x \in B$, $T^1_{k,x}(M, B)$ and $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathcal{G} \times L(\mathbb{R}^k, L\mathcal{G})$.

If we denote by $T^1_k(M, B) = \bigcup_{x \in B} T^1_{k,x}(M, B)$, then $(T^1_k(M, B), \tau, B)$ is a fiber bundle over B, with typical fiber $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathcal{G} \times L(\mathbb{R}^k, L\mathcal{G})$ and structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$.

The structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$ acts on the typical fiber $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathcal{G} \times L(\mathbb{R}^k, L\mathcal{G})$ through the left:

$$(A, g, \alpha)(a, g', u) = (Aa, gg', \alpha a + (L_g)_* u). \tag{2.2}$$

Here $(L_g)_*$ is the automorphism of the Lie algebra $L\mathcal{G}$ induced by the left translation L_g. The composition of the group $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$ is given by

$$(A, g, \alpha)(B, g', \beta) = (AB, gg', \alpha B + (L_g)_* \beta). \tag{2.3}$$

The neutral element is $(I_n, e, 0)$ and the inverse of an element (A, g, α) is given by $(A, g, \alpha)^{-1} = (A^{-1}, g^{-1}, -(L_g^{-1})_*\alpha A^{-1})$.

We may remark here that the multiplication law (2.3) can be defined also on $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$, and in this case $(L_g)_*$ is the automorphism of the Lie algebra $\mathcal{X}_c(F)$ induced by the left translation L_g, for an arbitrary element $g \in \mathrm{Diff}^{\infty}(F)$. With respect to this multiplication law we have that $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^k, \mathcal{X}_c(F))$ is also a group, where the multiplication law and the inverse are smooth, so it is a infinite dimensional Lie group. Consequently, we have that for the general case, a fibration (M, π, B, F) determine a fiber bundle $(T_k^1(M, B), \tau, B)$ over B with the typical fiber $L(\mathbb{R}^k, \mathbb{R}^n) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^k, \mathcal{X}_c(F))$ and the structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$.

An arbitrary Lie subgroup of $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ has the form $G_1 \times G_2 \times \Sigma$, where G_1 and G_2 are Lie subgroups of $\mathrm{Gl}(n, \mathbb{R})$ and, respectively, $\mathrm{Diff}^{\infty}(F)$ and Σ is a subset of $L(\mathbb{R}^n, \mathcal{X}_c(F))$.

Next we shall pay attention to the extreme cases when $k = 1$ and $k = n$. For $k = 1$ a first order fiber jet $J^1_{1,x}\tilde{c}$ is called a *fiber vector* at $x \in B$. The set of all fiber vectors at $x \in B$ is denoted by $T^1_{1,x}(M, B)$ and it is called the fiber tangent space. Any fibered cart (U, τ_U) at $x \in B$ induces a bijection between the fiber tangent space $T^1_{1,x}(M, B)$ and $\mathbb{R}^n \times \mathcal{G} \times L\mathcal{G}$.

For $k = n$ we consider first order fiber jets $J^1_{n,x}\tilde{\phi}$ of local fiber diffeomorphisms $\tilde{\phi}$ from a neighborhood of $\{0\} \times F$ in $\mathbb{R}^n \times F$ into a neighborhood of $\pi^{-1}(x)$ in M. Such a fiber jet is called a fiber frame at $x \in B$, the set of all first order fiber frames at x is denoted by $\mathcal{F}_x(M, B)$. This way we can construct a principal fiber bundle $\mathcal{F}(M, B) = \bigcup_{x\in B} \mathcal{F}_x(M, B)$ over the base manifold B with structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$. We call this principal fiber bundle the *fiber frame bundle* of the fibration (M, π, B, F). For a fiber bundle $(M, \pi, B, F, \mathcal{G})$ the corresponding fiber frame bundle has the structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$ which is a Lie subgroup of $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$.

A fiber G-structure of the fibration (M, π, B, F) (or of the fiber bundle $(M, \pi, B, F, \mathcal{G})$) is a reduction of the structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^{\infty}(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ (or $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$) to a Lie subgroup $G = G_1 \times G_2 \times \Sigma$. The corresponding fiber frame bundle to a G-structure will be denoted by $\mathcal{F}_G(M, B)$.

As an example, let us apply all the above considerations to the tangent bundle (TB, π, B) of a manifold B. The typical fiber is then $F = \mathbb{R}^n$ and the structural group is $\mathcal{G} = \mathrm{Gl}(n, \mathbb{R})$. Then the fiber tangent bundle $(T_1^1(\mathrm{TB}, B), \tau, B)$ has as typical fiber $\mathbb{R}^n \times \mathrm{Gl}(n, \mathbb{R}) \times M_n(\mathbb{R})$ and the structural group is $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Gl}(n, \mathbb{R}) \times L^2(\mathbb{R}^n, \mathbb{R}^n)$. Then the fiber frame bundle $\mathcal{F}(\mathrm{TB}, B)$ has as typical fiber and structural group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Gl}(n, \mathbb{R}) \times L^2(\mathbb{R}^n, \mathbb{R}^n)$ and consequently it is isomorph to the second order nonholonomic frame bundle $F^2(B)$ of the manifold B. Consequently, a fiber G-structure $\mathcal{F}_G(\mathrm{TB}, B)$ is then a second order nonholonomic G-structure, where $G = G_1 \times G_2 \times \Sigma$, G_1 and G_2 being Lie subgroups of $\mathrm{Gl}(n, \mathbb{R})$ and Σ is a subset of $L^2(\mathbb{R}^n, \mathbb{R}^n)$. In Section 2.4 we shall see that the integrability of a fiber G-structure is equivalent to the integrability of the corresponding second order nonholonomic G-structure.

2.3. The Lie groupoid of fiber jets

Let us consider (M, π, B, F) a fibration. For a local diffeomorphism $\tilde{\phi}$ of M that preserves the fiber structure we denote by ϕ its projected local diffeomorphism of the base manifold B. For any pair of points x_1, x_2 of B, denote by $C_{x_1,x_2}(M)$ the set of all local diffeomorphisms $\tilde{\phi}$ of M that preserves the fiber structure and such that its projection ϕ maps x_1 to x_2.

Then we say that $\tilde{\phi}, \tilde{\psi} \in C_{x_1,x_2}(M)$ determine the same first order fiber jet if there is a local diffeomorphism $\tilde{\kappa}$ from a neighborhood of $\{0\} \times F$ in $\mathbb{R}^n \times F$ into a neighborhood of $\pi^{-1}(x_1)$ in M that preserves the fiber structure and the maps $\tilde{\phi} \circ \tilde{\kappa}$ and $\tilde{\psi} \circ \tilde{\kappa}$ have the same first order fiber jet as we have defined it in the above section. Then, according to the previous section this means that $\tilde{\phi} \circ \tilde{\kappa} \sim_{1,x_2} \tilde{\psi} \circ \tilde{\kappa}$. It is easy to see that the above definition is independent of the local diffeomorphism $\tilde{\kappa}$. We shall denote by $J^1_{x_1,x_2}\tilde{\phi}$ the first order fiber jet of a local fiber diffeomorphism that maps $\pi^{-1}(x_1)$ into $\pi^{-1}(x_2)$. Consider now the union of all collections of fiber jets of fiber morphisms that map x_1 to x_2 for all possible pairs x_1, x_2 of B. We denote this union by $J^1(M, B)$ and we call it the space of first order fiber jets of M. This space has two canonical projection maps. The first projection is $\alpha : J^1(M, B) \to B$ and points at the source x_1 of a fiber jet $J^1_{x_1,x_2}\tilde{\phi}$, while the second projection $\beta : J^1(M, B) \to B$ points to the target point x_2.

Consider $J^1_{x_1,x_2}\tilde{\phi}$ and $J^1_{x_2,x_3}\tilde{\psi}$ two first order fiber jets such that $\alpha(J^1_{x_2,x_3}\tilde{\psi}) = \beta(J^1_{x_1,x_2}\tilde{\phi}) = x_2$. Then, we define the product of the fiber jets through

$$J^1_{x_2,x_3}\tilde{\psi} \cdot J^1_{x_1,x_2}\tilde{\phi} = J^1_{x_1,x_3}(\tilde{\psi} \circ \tilde{\phi}). \tag{2.4}$$

Theorem 2.1. *The set $J^1(M, B)$ of first order fiber jets of M with the canonical projections α and β and the multiplication law defined in (2.4) has a canonical structure of Lie groupoid over the base manifold B.*

Proof. We have to check first that all axioms for a groupoid are satisfied. We refer the reader to [12] for a good reference on groupoids; see also [2]:

(i) According to the definition of the multiplication law (2.4) we have that for two elements $Z, Z' \in J^1(M, B)$, the product $Z \cdot Z'$ is defined if and only if $\alpha(Z) = \beta(Z')$ and then $\beta(Z \cdot Z') = \beta(Z)$ and $\alpha(Z \cdot Z') = \alpha(Z')$.

(ii) The triple product $Z \cdot (Z' \cdot Z'')$ is defined if and only if $(Z \cdot Z') \cdot Z''$ is also defined and, when one of them is defined, the associative law $Z \cdot (Z' \cdot Z'') = (Z \cdot Z') \cdot Z''$ holds.

(iii) For each $x \in B$ there exists an element $1_x = J^1_{x,x}\tilde{\phi}$, $\tilde{\phi}$ is the identity map of a neighborhood of $\pi^{-1}(x)$, such that:

- $\alpha(1_x) = \beta(1_x) = x$;
- if $Z \cdot 1_x$ is defined then $Z \cdot 1_x = Z$;
- if $1_x \cdot Z$ is defined then $1_x \cdot Z = Z$.

(iv) For each $Z = J^1_{x_1,x_2}\tilde{\phi} \in J^1(M, B)$, there exists $Z^{-1} = J^1_{x_1,x_2}\tilde{\phi}^{-1} \in J^1(M, B)$ such that $Z^{-1} \cdot Z = 1_{x_1}$ and $Z \cdot Z^{-1} = 1_{x_2}$.

So, we proved that $J^1(M, B)$ is a groupoid over B.

Now, we introduce a C^∞-differentiable structure on $J^1(M, B)$ as follows.

For every pair of points $x_1, x_2 \in B$, let (U_1, τ_{U_1}) and (U_2, τ_{U_2}) be two fibered charts of the fibered bundle such that $x_1 \in U_1$ and $x_2 \in U_2$. The pair of fibered charts (U_1, τ_{U_1}) and (U_2, τ_{U_2}) induces a bijection between the subset $\alpha^{-1}(U_1) \cap \beta^{-1}(U_2)$ of $J^1(M, B)$ and $U_1 \times U_2 \times \mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$, the bijection being given by

$$J^1_{\tilde{x}_1,\tilde{x}_2}\tilde{\phi} \mapsto (\tilde{x}_1, \tilde{x}_2, J^1_{\tilde{x}_1,\tilde{x}_2}(\tau_{U_2,\tilde{x}_2} \circ \tilde{\phi} \circ \tau^{-1}_{U_1,\tilde{x}_1})) \quad \forall \tilde{x}_1 \in U_1, \ \ \tilde{x}_2 \in U_2. \tag{2.5}$$

The above defined bijection transfers the differentiable structure of $U_1 \times U_2 \times \mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ to $\alpha^{-1}(U_1) \cap \beta^{-1}(U_2)$. This way, $J^1(M, B)$ became a differentiable manifold. Now it is easy to see that the maps $\alpha, \beta : J^1(M, B) \to B$ are submersions. As $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ is a Lie group, then according to (2.5), the multiplication law (2.4) of $J^1(M, B)$ and the map $Z \in J^1(M, B) \mapsto Z^{-1} \in J^1(M, B)$ are compatible with the differentiable structure of $J^1(M, B)$. Consequently, we have that $J^1(M, B)$ is a differentiable groupoid over B. Also, we have from (2.5) that the map $\alpha \times \beta : J^1(M, B) \to B \times B$, given by $(\alpha \times \beta)(Z) = (\alpha(Z), \beta(Z))$ is a submersion, so the differentiable groupoid $J^1(M, B)$ is a Lie groupoid. More than that, as $\alpha \times \beta$ is surjective, the Lie groupoid $J^1(M, B)$ is a transitive Lie groupoid. □

Choose $x \in B$ arbitrarily and define

$$J^1_x(M, B) = \{Z \in J^1(M, B), \alpha(Z) = x\} \quad \text{and}$$
$$G(x) = \{Z \in J^1(M, B), \alpha(Z) = \beta(Z) = x\}.$$

Proposition 2.2. *$G(x)$ is a Lie group and $J^1_x(M, B)$ is a principal fiber bundle over B with structural group $G(x)$ and projection β.*

Proof. According to (2.4) and (2.5), for a fixed fiber chart (U, τ_U) at $x \in B$, we have that the map:

$$Z \in G(x) \mapsto J^1_{x,x}\tau_{U,x} \circ Z \circ J^1_{x,x}\tau^{-1}_{U,x} \in \mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F)) \tag{2.6}$$

is a bijection that transfers the structure of the Lie group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ to $G(x)$ such that the map (2.6) becomes an isomorphism of Lie groups.

As $J^1_x(M, B) = \alpha^{-1}(x)$ from (2.5) we can see that if we fix $\tilde{x}_1 = x$ then we have a map from $\beta^{-1}(U_2)$ to $U_2 \times \mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$, that is a bijection. If we consider the set of all such maps when U_2 covers B, we have an atlas of fibered charts for a principal fibered structure on $J^1_x(M, B)$ over B with the projection β. The above considered maps from $\beta^{-1}(U_2)$ to $U_2 \times \mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$ are local trivializations. □

Remark 2.3. For an element $x \in B$, the set $G(x) = \alpha^{-1}(x) \cap \beta^{-1}(x)$ consists of first order fiber jets of all local diffeomorphisms of a neighborhood of $\pi^{-1}(x)$, so this is diffeomorphic to the Lie group $\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Diff}^\infty(F) \times L(\mathbb{R}^n, \mathcal{X}_c(F))$, the diffeomorphism between these two Lie groups being given by (2.6). The group $G(x)$ is called the isotropy group at $x \in B$ of the groupoid $J^1(M, B)$.

If we have a fiber bundle $(M, \pi, B, F, \mathcal{G})$, the isotropy group $G(x)$ at $x \in B$ of the Lie groupoid $J^1(M, B)$ is diffeomorphic with the Lie group $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$.

Remark 2.4. The principal fiber bundle $J^1_x(M, B)$ from the above proposition is diffeomorphic to the fiber G-structure $\mathcal{F}_G(M, B)$ we introduced in Section 2.2, where $G = \mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$.

2.4. Integrability conditions for a fiber G-structure

Consider now a fiber bundle $(M, \pi, B, F, \mathcal{G})$ and a fiber G-structure $\mathcal{F}_G(M, B)$. This is a principal fiber bundle with structural group $G = G_1 \times G_2 \times \Sigma$, the projection β and the base manifold B, where G_1 is a Lie subgroup of $\mathrm{Gl}(n, \mathbb{R})$, G_2 a Lie subgroup of $\mathcal{G}$ and Σ a subset of $L(\mathbb{R}^n, L\mathcal{G})$.

For the trivial fiber bundle $(\mathbb{R}^n \times F, \mathrm{pr}_1, \mathbb{R}^n, F, \mathcal{G})$ its principal fiber bundle $\mathcal{F}(\mathbb{R}^n \times F, \mathbb{R}^n)$ is isomorphic to $R^n \times \mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$. By this isomorphism we can transport a Lie subgroup $G = G_1 \times G_2 \times \Sigma$ of $\mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$ to obtain a fiber G-reduction of $\mathcal{F}(\mathbb{R}^n \times F, \mathbb{R}^n)$. The fiber G-structure we defined above is called the *flat* (*or integrable*) *G-structure* on $\mathbb{R}^n$ and we denote it by $\mathcal{F}_G(\mathbb{R}^n)$.

Definition 2.5. A fiber G-structure $\mathcal{F}_G(M, B)$ is said to be integrable if it is locally isomorphic to the flat G-structure on $\mathbb{R}^n$.

Theorem 2.6. *A fiber G-structure $\mathcal{F}_G(M, B)$ is integrable if and only if around every point in B, there is a fibered chart (U, τ_U) of the fibered structure $(M, \pi, B, F, \mathcal{G})$ such that*

$$J^1\tau_U^{-1} : x \in U \to J^1_{x,x}\tau_U^{-1} \tag{2.7}$$

is a local section of the principal fiber bundle $\mathcal{F}_G(M, B)$.

Proof. According to the proof of Proposition 2.2 for every point in B, a fibered chart (U, τ_U) of the fibered bundle (M, π, B) determines a fibered chart

$$\begin{aligned} &Z_x \in \beta^{-1}(x) \subset \beta^{-1}(U) \\ &\quad \mapsto J^1_{x,x}\tau_{U,x} \circ Z_x \circ J^1_{x,x}\tau_{U,x}^{-1} \in U \times \mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G}). \end{aligned} \tag{2.8}$$

Now, if we assume that for every point in B, there is a fibered chart (U, τ_U) such that $Z = J^1\tau_U^{-1} : x \in U \mapsto Z_x = J^1_{x,x}\tau_{U,x}^{-1} \in G(x)$ is a local section of $\mathcal{F}_G(M, B)$, then (2.8) is an isomorphism between $\beta^{-1}(U)$ and $U \times \mathrm{Gl}(n, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^n, L\mathcal{G})$. Using a differentiable partition of unity, we can construct from these a local isomorphism between the fiber G-structure $\mathcal{F}_G(M, B)$ and the flat G-structure on $\mathbb{R}^n$.

Conversely, suppose that the fiber G-structure is integrable. So for every point in B, there is an open set $U \subset B$ and an isomorphism of fiber G-structures:

$$\phi_U : \beta^{-1}(U) \to \chi(U) \times G_1 \times G_2 \times \Sigma,$$

where (U, χ) is a local chart of the base manifold B. We have also that $x \in B \mapsto (x, I_n, e, 0)$ is a section of the flat G-structure $\mathcal{F}_G(\mathbb{R}^n)$ over $\chi(U)$. By composition with ϕ_U^{-1} we get a local section of $\mathcal{F}_G(M, B)$ over U that is the first order fiber jet of a fibered map τ_U^{-1} : $U \times F \to \pi^{-1}(U)$. □

A local section of the principal fiber frames $\mathcal{F}(M, B)$ (or a local field of fiber frames on B) can be expressed in local coordinates as

$$P(x) = (G^i_j(x), y^a(x, \cdot), L^a_j(x, \cdot)) \quad \forall x \in U \subset B. \tag{2.9}$$

Here $G^i_j(x)$ is a field of frames on U, $(y^a(x, \cdot))_{a=\overline{1,m}}$ are the m-components of a diffeomorphism of the fiber F_x and for all $j = \overline{1, m}$, $(L^a_j(x, \cdot))$ are m-vector fields with compact support on F. For a field of fiber frames P on B, we can consider also the inverse of the field, P^{-1}, defined by

$$P^{-1}(x) = \left((G^{-1})^j_k(x), \tilde{y}^b(x, \cdot), (L^{-1})^b_k := -\frac{\partial \tilde{y}^b}{\partial y^a} L^a_i \frac{\partial \tilde{x}^i}{\partial x^j}\right). \tag{2.10}$$

Theorem 2.7. *A fiber G-structure $\mathcal{F}_G(M, B)$ is integrable if and only if for any point in B there is an open set $U \subset B$ and a field of fiber G-frames on U*

$$P(x) = (G^i_j(x), y^a(x, \cdot), L^a_j(x, \cdot))$$

for which the following tensors vanish:

$$\frac{1}{2} G^i_j(x) \left(\frac{\partial (G^{-1})^j_k}{\partial x^l} - \frac{\partial (G^{-1})^j_l}{\partial x^k}\right), \qquad \frac{1}{2} \frac{\partial y^a}{\partial \tilde{y}^b} \left(\frac{\partial (L^{-1})^b_k}{\partial x^l} - \frac{\partial (L^{-1})^b_l}{\partial x^k}\right) \quad \text{and}$$
$$\frac{\partial y^a}{\partial \tilde{y}^b} \left(\frac{\partial (L^{-1})^b_j}{\partial y^c} - \frac{\partial^2 \tilde{y}^b}{\partial y^c \partial x^j}\right). \tag{2.11}$$

Proof. According to Theorem 2.6 we have that the fiber G-structure $\mathcal{F}_G(M, B)$ is integrable if and only if for the field of fiber frames P there is a fibered chart (U, τ_U) such that

$$P(x) = J^1_{x,x} \tau_U^{-1} \quad \forall x \in U.$$

If the fibered map $\tau_U : (x^i, y^b) \in \pi^{-1}(U) \mapsto (\tilde{x}^i, \tilde{y}^a) \in U \times F$ has the equations

$$\tilde{x}^i = \tilde{x}^i(x^j), \quad \operatorname{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \qquad \tilde{y}^a = \tilde{y}^a(x^j, y^b), \quad \operatorname{rank}\left(\frac{\partial \tilde{y}^a}{\partial y^b}\right) = m,$$

then we have to prove that the tensors (2.11) vanish if and only if

$$(G^{-1})^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}, \qquad (L^{-1})^a_j = -\frac{\partial \tilde{y}^a}{\partial y^b} L^b_i \frac{\partial \tilde{x}^i}{\partial x^j}. \tag{2.12}$$

The tensors (2.11) are the torsion components of the complete parallelism (linear connection) D on $\pi^{-1}(U) \subset M$ induced by the following field of frames on $\pi^{-1}(U)$:

$$F(x, y) = \begin{pmatrix} G^i_j(x) & 0 \\ L^a_j(x, y) & \dfrac{\partial y^a}{\partial \tilde{y}^b}(x, y) \end{pmatrix}. \tag{2.13}$$

This means that $H_i = (G^{-1})_i^j(\partial/\partial\tilde{x}^j) + (L^{-1})_i^b(\partial/\partial\tilde{y}^b)$ and $V_a = (\partial\tilde{y}^b/\partial y^a)(\partial/\partial\tilde{y}^b)$ is a field of frames on $\pi^{-1}(U)$. Then let D be the unique linear connection for which the frame $\{H_i, V_a\}$ is covariant constant, that is $D_Z H_i = D_Z V_a = 0\,\forall Z$.

The linear connection D is curvature free. If we express the torsion $T(Z, W) = D_Z W - D_W Z - [Z, W]$ with respect to the frame $\{H_i, V_a\}$, then there are only three nonzero components and these are the tensors (2.11). □

Next we shall apply the theory developed in this section for the particular case when the fiber bundle (M, π, B) is the tangent bundle (TB, π, B). Let this be the case. As we have already seen in Section 2.2, the fiber frame bundle $\mathcal{F}(\mathrm{TB}, B)$ has as typical fiber and structural group $\mathrm{Gl}(n,\mathbb{R}) \times \mathrm{Gl}(n,\mathbb{R}) \times L^2(\mathbb{R}^n,\mathbb{R}^n)$. But $\mathrm{Gl}(n,\mathbb{R}) \times \mathrm{Gl}(n,\mathbb{R}) \times L^2(\mathbb{R}^n,\mathbb{R}^n)$ is the structural group of the second order nonholonomic frame bundle $F^2(B)$ of the manifold B. Consequently we have that the fiber frame bundle $\mathcal{F}(\mathrm{TB}, B)$ and the second order nonholonomic frame bundle $F^2(B)$ are diffeomorphic. Consider now $G = G_1 \times G_2 \times \Sigma$ a Lie subgroup of $\mathrm{Gl}(n,\mathbb{R}) \times \mathrm{Gl}(n,\mathbb{R}) \times L^2(\mathbb{R}^n,\mathbb{R}^n)$. A fiber G-structure $\mathcal{F}_G(\mathrm{TB}, B)$ determines a second order nonholonomic G-structure $F_G^2(B)$ and the integrability of one of these two implies the integrability of the other one.

Due to the particular form of the typical fiber $\mathbb{R}^n$, a local section of the principal frame bundle $\mathcal{F}(\mathrm{TB}, B)$ and hence a local section of the second order nonholonomic frame bundle $F^2(B)$ has the form

$$P(x) = (G^i_j(x), H^a_b(x), L^a_{bj}(x)).$$

The inverse of this local section is given by

$$P^{-1} = ((G^{-1})^j_k, (H^{-1})^b_c, (L^{-1})^b_{ck} := -(H^{-1})^b_a L^a_{di}(H^{-1})^d_c(G^{-1})^j_k).$$

Theorem 2.8 has the following version.

Theorem 2.8. *A fiber G-structure $\mathcal{F}_G(\mathrm{TB}, B)$ or the second order nonholonomic G-structure $F^2_G(B)$ is integrable if and only if for any point in B there is an open set $U \subset B$ and a field of fiber G-frames on U, $P(x) = (G^i_j(x), H^a_b(x), L^a_{bj}(x))$ for which the following tensors vanish*:

$$\frac{1}{2}G^i_j(x)\left(\frac{\partial(G^{-1})^j_k}{\partial x^l} - \frac{\partial(G^{-1})^j_l}{\partial x^k}\right), \qquad H^a_b\left((L^{-1})^b_{ck} - \frac{\partial(H^{-1})^b_c}{\partial x^k}\right). \tag{2.14}$$

Proof. Due to the particular form of the section P, the second tensor (2.11) vanishes if and only if the third one does. The tensors (2.13) vanish if and only if there exists a fibered diffeomorphism

$$\tilde{x}^i = \tilde{x}^i(x^j), \quad \mathrm{rank}\left(\frac{\partial\tilde{x}^i}{\partial x^j}\right) = n, \qquad \tilde{y}^a = H^a_b(x^j)y^b, \quad \mathrm{rank}(H^a_b) = n,$$

such that $G^j_i = \partial x^j/\partial\tilde{x}^i$ and $L^a_{bj} = \partial H^a_b/\partial x^j$. The fiber diffeomorphism (2.14) assure the integrability of the fiber G-structure $\mathcal{F}_G(\mathrm{TB}, B)$ or the second order nonholonomic G-structure $F^2_G(B)$. □

3. Part II. Bodies with microstructure

3.1. Configurations and first grade response for a body bundle

In continuum mechanics a material body B, that represents the *macromedium*, is represented by a three-dimensional manifold which can be covered with just one chart, $\kappa : B \to \mathbb{R}^3$. Such a chart is called a configuration of the macromedium. We assume that the body B has a smoothly distributed microstructure represented by a typical fiber F, that is an m-dimensional differentiable manifold. The geometric object that corresponds to this concept is a fiber bundle (M, π, B, F). For this we assume also that the body bundle M is globally trivializable. This means that there is a global fibered chart

$$\tau : M \to B \times F.$$

If we consider the composition of $\kappa \times \mathrm{Id}_F$ by τ we have a *configuration* of the body bundle:

$$\tilde{\kappa} = (\kappa \times \mathrm{Id}_F) \circ \tau : M \to \mathbb{R}^3 \times F.$$

When there is no danger of confusion, we shall identify the body bundle M with one of its configuration, $M \equiv \tilde{\kappa}_0(M) = \mathbb{R}^3 \times F$. Such a configuration is called a *reference configuration*. A change of configurations, or a *deformation* is defined then as the composition $\tilde{\kappa} \circ \tilde{\kappa}_0^{-1} : M \to \mathbb{R}^3 \times F$. If the reference configuration $\tilde{\kappa}_0$ is fixed, we refer to the deformation $\tilde{\kappa} \circ \tilde{\kappa}_0^{-1}$ through the configuration $\tilde{\kappa}$.

Let X^I, Y^A and x^i, y^a be the local coordinate systems in the body bundle and in the cross product $\mathbb{R}^3 \times F$. From now on, we adopt the convention that the indices $H, I, J, K, L, h, i, j, k, l$ vary within the range $1, 2, 3$, while the indices $A, B, C, D, E, a, b, c, d, e$ vary within the range $1, \ldots, m$. A deformation $\tilde{\kappa} \circ \tilde{\kappa}_0^{-1}$, or a configuration $\tilde{\kappa}$ of the body bundle is given by the $3 + m$ smooth functions:

$$x^i = x^i(X^I), \quad \mathrm{rank}\left(\frac{\partial x^i}{\partial X^I}\right) = 3, \quad y^a = y^a(X^I, Y^A), \quad \mathrm{rank}\left(\frac{\partial y^a}{\partial Y^A}\right) = m. \tag{3.1}$$

For any $x \in B$, $(\tilde{\kappa} \circ \tilde{\kappa}_0^{-1})(x, \cdot)$ is a diffeomorphism of the typical fiber F, that is $(\tilde{\kappa} \circ \tilde{\kappa}_0^{-1})(x, \cdot) \in \mathrm{Diff}^\infty(F)$. When a structural group $\mathcal{G}$ that acts on the typical fiber F is assigned, then we have to assume also that $(\tilde{\kappa} \circ \tilde{\kappa}_0^{-1})(x, \cdot) \in \mathcal{G}$. With these, the last m-equations of (3.1) can be written as $y^a = L_g(Y^A)$, where L_g is the left translation induced by an element $g \in \mathrm{Diff}^\infty(F)$ (or $g \in \mathcal{G}$).

Our attention now is focused on bundle bodies whose mechanical behavior is local. This means that the deformation evaluated outside an arbitrarily small neighborhood of each point of B does not affect the material response at the point. In particular we consider the case when the material response is of the first grade that is it involves only the values of the first derivative of the configuration with respect to the base manifold coordinates.

For a configuration $\tilde{\kappa}$ with Eqs. (3.1) we consider its first order fiber jet

$$J^1_X \tilde{\kappa} = (x^i_{,J}(X^I), y^a(X^I, \cdot), y^a_{,J}(X^I, \cdot))$$

with commas denoting partial derivatives.

An elastic behavior of a body bundle M is completely characterized by an *energy functional* W. This energy functional is assumed to be given as the integral over the macromedium B of an *energy density functional* w whose independent argument is the first order fiber jet of the configuration. If $\mathrm{d}V$ is the volume-form in $\mathbb{R}^3$, then we may write the energy functional as

$$W = \int_{\kappa_0(B)} w(J^1_X\tilde{\kappa}; X)\,\mathrm{d}V(X). \tag{3.2}$$

We note that the energy density w is still a functional as far as its independence on the functions $y^a(X^I, \cdot)$ and their X-derivatives is concerned. More precisely, the value of w depends on the values of x^i and their derivatives at X^I and depends also on the functions $y^a(X^I, \cdot)$ and their derivatives as functions of the fiber coordinates Y^A. So, the behavior of the body bundle is local only with respect to the dependence of the deformation of the micromedium, but it may be global in terms of its dependence of the deformation of the micromedium. To emphasize this fact we can write in coordinates the energy density w as

$$w = w(x^i_{,J}(X^I), y^a(X^I, \cdot), y^a_{,J}(X^I, \cdot); X^I). \tag{3.3}$$

3.2. Uniformity and material symmetries

As we can see from the formula (3.3) the energy density w varies from point to point of the macromedium B, by the dependence of w on the last argument, X. We can say then that the two points X_1 and X_2 of B are *materially isomorphic* (read "made of the same material") if there exists a body bundle diffeomorphism $\tilde{\kappa}_{1,2}$ such that $\kappa_{1,2}(X_1) = X_2$ and

$$w(J^1_{X_2}\tilde{\kappa} \circ J^1_{X_1}\tilde{\kappa}_{1,2}; X_1) = w(J^1_{X_2}\tilde{\kappa}; X_2) \tag{3.4}$$

for all configurations $\tilde{\kappa}$. Next, we shall use the notation $P(X_1, X_2) = J^1_{X_2}\tilde{\kappa}_{1,2}$. Physically speaking, this jet ("material isomorphism") represents a "transplant operation" that achieves a perfect graft as far as the mechanical behavior is concerned. What has been done is to cut out a first order neighborhood of the point X_1 including the fiber $\pi^{-1}(X_1)$ deform it according to the map $P(X_1, X_2)$, and implant it into the place of a similar neighborhood of X_2 and its fiber. The identity (3.4) expresses that the graft has been successful, and this can only happen if the materials are the same.

Definition 3.1. A body (M, π, B) with microstructure is said to be *materially uniform* if all points of B are pairwise materially isomorphic.

Consider now $J^1(M, B)$ the Lie groupoid, with the projections $\alpha, \beta : J^1(M, B) \to B$, introduced in Section 2.3. For a given uniform body bundle M with energy density w, we can consider the set of all first order fiber jets representing material isomorphisms, and we denote it by $J^1_w(M, B)$. We have that $J^1_w(M, B)$ is a subgroupoid of $J^1(M, B)$. Our first assumption now is that $J^1_w(M, B)$ is a differentiable subgroupoid of $J^1(M, B)$.

Remark 3.2. For a body bundle M, the property of uniformity is equivalent to the transitivity of $J^1_w(M, B)$.

According to [12] a differentiable groupoid that is transitive is a Lie groupoid. Our second assumption now is that the $J^1_w(M, B)$ is a Lie subgroupoid of $J^1(M, B)$. In this case the map $\alpha \times \beta : J^1_w(M, B) \to B \times B$ is a surjective submersion, and hence, there exist local sections. A local section:

$$\begin{aligned} P &: (X_1, X_2) \in U_1 \times U_2 \subset B \times B \\ &\mapsto P(X_1, X_2) \in \alpha^{-1}(X_1) \cap \beta^{-1}(X_2) \subset J^1_w(M, B) \end{aligned} \tag{3.5}$$

is called a local material uniformity. In this case we say that the body bundle enjoys local uniformity. We may conclude now with the following proposition.

Proposition 3.3. *A body bundle* (M, π, B) *is uniform if and only if the associated groupoid* $J^1_w(M, B)$ *is a Lie groupoid.*

A body bundle diffeomorphism that maps a point X of the macromedium B to itself may happen to have the property that it leaves the material response at X unchanged. For such a diffeomorphism, its first order fiber jet will define a material symmetry at X.

Definition 3.4. For a body bundle (M, π, B) a *material symmetry* at $X \in B$ is a first order fiber jet $J^1_X\Psi$, where Ψ is a local fiber diffeomorphism of M at X and for which the following identity is true:

$$w(J^1_X\tilde{\kappa}; X) = w(J^1_X\tilde{\kappa} \circ J^1_X\Psi; X)$$

for all configurations $\tilde{\kappa}$.

We denote by $G(X)$ the set of all material symmetries at X. It is easy to see that $G(X)$ is a group with the composition of jets which is called the *group of material symmetries* (or the *isotropy group*) at X. We have also that

$$G(X) = \alpha^{-1}(X) \cap \beta^{-1}(X) = \{Z \in J^1_w(M, B); \alpha(Z) = \beta(Z) = X\}.$$

A material symmetry at a point $X \in B$ is given by a triple $(G(X), L_g(X, \cdot), L(X, \cdot))$. Here G is an element of the linear group $\mathrm{Gl}(3, \mathbb{R})$ and has the meaning of a symmetry of the macromedium B at X. L_g is the left translation induced by an element g of the structural group $\mathcal{G}$ that acts on the fiber $\pi^{-1}(X)$ and $L(X, \cdot)$ is a mixed symmetry of micro- and macrostructure. So, we have that the material symmetry group is a subgroup of the semidirect product $\mathrm{Gl}(3, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^3, L\mathcal{G})$, where the multiplication law is, according to (2.3), given by

$$(G, L_g, L) \cdot (G', L_{g'}, L') = (GG', L_{gg'}, LG' + (L_g)_*L'). \tag{3.6}$$

The neutral element of this group is $(I_3, e, 0)$, where e is the neutral element of the structural group $\mathcal{G}$. The inverse of an element (G, L_g, L) is given by $(G, L_g, L)^{-1} = (G^{-1}, L_g^{-1}, -(L_g^{-1})_*LG^{-1})$.

3.3. Homogeneous bodies with microstructure

An equivalent way to define the uniformity for a body bundle is to verify the existence of the material isomorphisms between points of B and a fixed point $X_0 \in B$. Let us fix an *archetypal point* $X_0 \in B$, but we think this archetypal point as a point carrying the typical fiber and a first order neighborhood of both. Consider

$$G(X_0) = \{Z \in J^1_w(M, B); \alpha(Z) = \beta(Z) = X_0\} = \alpha^{-1}(X_0) \cap \beta^{-1}(X_0) \quad \text{and}$$
$$J^1_{X_0,w}(M, B) = \{Z \in J^1_w(M, B); \alpha(Z) = X_0\} = \alpha^{-1}(X_0).$$

According to Proposition 2.2 we have the following proposition.

Proposition 3.5.

(1) *The material symmetry group $G(X_0)$ is a Lie group, diffeomorphic to a Lie subgroup of* $\mathrm{Gl}(3, \mathbb{R}) \times \mathcal{G} \times L(\mathbb{R}^3, L\mathcal{G})$.
(2) $J^1_{X_0,w}(M, B)$ *is a principal fiber bundle over B with structural group $G(X_0)$ and projection β.*

A body bundle is locally uniform if the principal fiber bundle $J^1_{X_0,w}(M, B)$ admits local sections:

$$P : X \in U \subset B \mapsto P(X) = P(X_0, X) \in \beta^{-1}(X) \subset J^1_{X_0,w}(M, B).$$

Let (x^i) be the coordinates of the archetypal point X_0 and y^a the fiber coordinates along the typical fiber. A uniformity field can be written then as

$$P(X) = (G^I_i, Y^A(X, \cdot), L^A_i(X, \cdot)).$$

Here $Y^A(X, \cdot)$ and $L^A(X, \cdot)$ are functions of y^a.

We have seen in the previous section that the concept of uniformity was introduced to translate the idea that “all points of the body bundle are made of the same material”. A stronger concept than this is the concept of homogeneity which means that “there exists a reference configuration in which the energy density functional w is independent of position”. Of course a homogeneous body bundle is uniform (there are local versions for these concepts with the same implication). In a more precise way we introduce the following definition.

Definition 3.6. A body bundle (M, π, B) is said to be homogeneous with respect to a given fiber frame of an archetypal point X_0 if it admits a global deformation $\tilde{\kappa}$ such that the first order fiber jet $J^1_X \tilde{\kappa}^{-1} = P(X)$ is a uniformity field.

This is equivalent to say that the first order fiber jet $J^1_X \tilde{\kappa}^{-1}$ is a section of the principal fiber bundle $J^1_{X_0,w}(M, B)$.

According to Theorem 2.6 we have the following result.

Theorem 3.7. *The body bundle (M, π, B) is homogeneous if and only if the associated principal fiber bundle $J^1_{X_0,w}(M, B)$ is integrable.*

Let us consider now a uniformity field $P(X) = (G_i^I(X), Y^A(X,\cdot), L_i^A(X,\cdot))$ and denote by $P^{-1}(X) = (G_I^i(X), y^a(X,\cdot), L_I^a(X,\cdot))$ its inverse, given by (2.10). Then if we take into account Theorem 2.8 we have the following result for the homogeneity of a body bundle.

Theorem 3.8. *A body bundle* (M, π, B) *is homogeneous if and only if there exists a uniformity field* $P(X) = (G_i^I(X), Y^A(X,\cdot), L_i^A(X,\cdot))$ *for which the following tensors vanish*:

$$\frac{1}{2}G_i^I\left(\frac{\partial G_J^i}{\partial X^K} - \frac{\partial G_K^i}{\partial X^J}\right), \qquad \frac{1}{2}\frac{\partial Y^A}{\partial y^a}\left(\frac{\partial L_J^a}{\partial X^K} - \frac{\partial L_K^a}{\partial X^J}\right) \quad \textit{and}$$
$$\frac{\partial Y^A}{\partial y^a}\left(\frac{\partial L_J^a}{\partial Y^B} - \frac{\partial^2 y^a}{\partial Y^B \partial X^J}\right). \tag{3.7}$$

The tensors (3.7) are called the *inhomogeneity tensors*. The first tensor (3.7) measures the inhomogeneity of the microstructure.

If we apply the theory developed in this section for the particular case when the body bundle (M, π, B) is the tangent bundle (TB, π, B) we recover the theory of second order materials, as it was developed in [2]. In this case we say that the body B has a linear microstructure.

As the typical fiber in this case is $F = \mathbb{R}^3$ then the functions $Y^A(X,\cdot)$ and $L_i^A(X,\cdot)$ that appear in a uniformity field are linear with respect to the fiber coordinates (y^a) we can write then a uniformity field as

$$P(X) = (G_i^I(X), H_a^A(X), L_{ai}^A(X)).$$

The inverse of this uniformity field is denoted by $P^{-1} = (G_I^i, H_A^a, L_{AI}^a)$. As the second inhomogeneity tensor (3.7) vanishes if and only if the third one does, we have the following result that correspond to Theorem 3.8.

Theorem 3.9. *A bundle B with linear microstructure is homogeneous if and only if there exists a uniformity field* $P(X) = (G_i^I(X), H_a^A(X), L_{ai}^A(X))$ *for which the following inhomogeneity tensors vanish*:

$$\frac{1}{2}G_i^I\left(\frac{\partial G_J^i}{\partial X^K} - \frac{\partial G_K^i}{\partial X^J}\right), \qquad H_a^A\left(L_{BJ}^a - \frac{\partial H_B^a}{\partial X^J}\right). \tag{3.8}$$

The inhomogeneity tensors (3.8) have been discovered in [9], the first one is the torsion of a complete parallelism on the base manifold B, while the second one appears as the difference of two parallelisms (linear connections) on B.

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References

[1] M. Epstein, I. Bucataru, Continuous distributions of dislocations in bodies with microstructure, J. Elasticity 70 (2003) 237–254.

[2] M. Epstein, M. de León, Geometrical theory of uniform Cosserat media, J. Geom. Phys. 26 (1998) 127–170.

[3] M. Epstein, M. de León, Homogeneity conditions for generalized Cosserat media, J. Elasticity 43 (3) (1996) 189–201.

[4] M. Epstein, M. de León, Geometric characterization of the homogeneity of media with microstructure, Extracta Math. 11 (1996) 116–126.

[5] M. Epstein, M. de León, Geometrical theory of uniform Cosserat media, J. Geom. Phys. 26 (1998) 127–170.

[6] M. Epstein, M. de León, Continuous distributions of inhomogeneities in liquid-crystal-like bodies, Proc. R. Soc. London A 457 (2001) 2507–2520.

[7] K. Kondo, Geometry of elastic deformation and incompatibility, in: Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry, Tokyo Gakujutsu Benken Fukyu-Kai, 1955.

[8] E. Kroener, Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen, Arch. Rat. Mech. Anal. 4 (1960) 273–334.

[9] M. de León, M. Epstein, On the integrability of second order G-structures with applications to continuous theories of dislocations, Rep. Math. Phys. 33 (3) (1993) 419–436.

[10] M. de Leon, M. Epstein, Material bodies of higher grade, C.R. Acad. Sci. Paris, Ser. I 319 (1994) 615–620.

[11] M. de León, M. Epstein, The geometry of uniformity in second-grade elasticity, Acta Mech. 114 (1996) 217–224.

[12] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, Cambridge University Press, Cambridge, 1987.

[13] P.W. Michor, Gauge Theory for Fiber Bundles, Bibliopolis, Napoli, 1991.

[14] R. Miron, M. Anastasiei, Vector Bundles and Lagrange Spaces with Applications to Relativity, Geometry Balkan Press, Bucharest, 1997.

[15] W. Noll, Materially uniform simple bodies with inhomogeneities, Arch. Rat. Mech. Anal. 27 (1967) 1–32.

[16] J. Slovac, Smooth structures on fibre jet spaces, Czech. Math. J. 36 (111) (1986) 358–375.

[17] C.C. Wang, On the geometric structure of simple bodies: a mathematical foundation for the theory of continuous distributions of dislocations, Arch. Rat. Mech. Anal. 27 (1967) 33–94.